Representing the Functions $Re(\cdot)$ and $Im(\cdot)$ as Complex Numbers

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Necessary Background

Complex numbers, such as z, are ordered pairs of real numbers such that

$$z = (x, y). \tag{1}$$

Explicitly, x and y are real numbers, $x, y \in \mathbb{R}$, with x being the "real" component of z and y being the "imaginary" component. (Note: y is still real, as stated above.) Functions exist to separate z into their constituents x and y, namely $Re(\cdot)$ and $Im(\cdot)$. $Re(\cdot)$ gives the real part of a complex number, while $Im(\cdot)$ gives the complex part of a complex number such that

$$Re(z) = x \quad \text{and} \quad Im(z) = y. \tag{2}$$

An introduction to complex numbers (no matter how brief) would not be complete with defining the pair $(0, 1) = i$, called the imaginary unit.

Finally, it will be useful to define some common quantities related to a complex number z. First, the "modulus" or absolute value of a complex number is defined as

$$|z| = \sqrt{x^2 + y^2} \tag{3}$$

for $z = (x, y)$. Second, the "conjugate" of a complex number z is given by

$$\bar{z} = (x, -y) \tag{4}$$

meaning the sign of the imaginary coefficient is reversed.

Motivation

Though $Re(\cdot)$ and $Im(\cdot)$ are often called functions, defined such that their usual properties (from Equation (2)) and outputs hold, the form of these functions is not clear. However, from their definition, both functions take an input from the set of complex numbers ($\mathbb{C}$) and generate an output in the real numbers ($\mathbb{R}$). One existing for formulation takes

$$Re(z) = \frac{z + \bar{z}}{2} \quad \text{and} \quad Im(z) = \frac{z - \bar{z}}{2i} \tag{5}$$

but the approach here will be to find complex numbers that when multiplied with z act as $Re(\cdot)$ and $Im(\cdot)$. No application for such a formulation is currently known, but the process is inherently amusing, and could have niche applications.

Formulation of the Desired Functions

The intent is to formulate two different complex numbers such that their multiplication with the complex input (z) yields $Re(z) = x$ and $Im(z) = y$, respectively. First, the real version of such a number is defined as

$$z_r = (x_r, y_r) \quad \text{such that} \quad z_r z = Re(z) = x. \tag{6}$$

The corresponding complex number is defined as

$$z_c = (x_c, y_c) \quad \text{such that} \quad z_c z = Im(z) = y. \tag{7}$$

Since x and y are real numbers, the equations

$$z_r z = (x, 0) \quad \text{and} \quad z_c z = (y, 0) \tag{8}$$

can be solved for z_r and z_c, respectively. This can be done by division (output/z) or by solving the simultaneous equations produced in each case using the multiplication of complex numbers as usual. Regardless of method, the two complex numbers of interest are found to be

$$z_r = (x_r, y_r) = \left(\frac{x^2}{|z|^2}, -\frac{xy}{|z|^2} \right) \quad \text{and} \quad z_c = (x_c, y_c) = \left(\frac{xy}{|z|^2}, -\frac{y^2}{|z|^2} \right), \tag{9}$$

which satisfactorily operate as $\Re(\cdot)$ and $\Im(\cdot)$ by experiment.