An Alternative Method to Compute Gauss's Sum

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A common story in mathematics is that of Carl Freiderich Gauss finding the general formula for the finite series

$$\sum_{i=1}^{n} i \tag{1}$$

as a response to his teacher's request to sum the numbers from 1 to 100 inclusive... at age 10.

Gauss's Original Method

His original method is presented here to serve as a comparison to the proposed alternative method. The value of this sum (called S) may be written in expanded form as

$$\begin{array}{ccccccccccc} \mathrm{S} & = & 1 & + & 2 & + & 3 & + & ... & + & \mathrm{n} & , \\ \mathrm{S} & = & \mathrm{n} & + & \text{(n-1)} & + & \text{(n-2)} & + & ... & + & 1 & , \end{array} \tag{2}$$

where both lines above are equivalent, but the second has leveraged the commutative property of addition to reverse the order of the terms on the right side. Now taking the sum of the two lines in Equation (2), being careful to sum the ith term on top with the ith term below

$$2S = (n+1) + (n+1) + (n+1) + ... + (n+1). \tag{3}$$

Since there were originally n terms in the original sum, the sum above (when arranged this way) has n terms which are each equal to $(n+1)$. Clearly, the right side is then equal to $n(n+1)$, and after dividing by the 2 on the left, Gauss's sum is computed as

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}. \tag{4}$$

There is no doubt that this method is beautiful and simple. However, this method may not be obvious or intuitive to everybody, and there is of course no harm in exploring a secondary method. [1] Further, a method which arises in the course of the secondary proof looks to be promising for some series computations (if it is not common knowledge already).

Alternative Method

The general overview of the alternative method is:

- construct a more general form of Gauss's sum (which begins at an arbitrary number instead of one)
- rearrange the above sum so it contains Gauss's sum as one of its terms
- examine a limit in which the original sum is known in order to simplify the rearranged expression
- solve explicitly for Gauss's sum.

[1] Indeed, this beautiful and simple method was not at all obvious when I revisited the problem myself. Instead my efforts developed into the alternative method which is the focus of this document.

One could liken this method to that of solving of a differential equation in the sense that a specific initial/boundary value is necessary to compute the sum of interest, though this connection is admittedly tenuous. Though this method is more involved than Gauss's method, none of the steps are necessarily more complicated, but some do use slightly more advanced mathematics than Gauss uses, though nothing outside of an undergraduate calculus sequence. [2]

The more general series of interest in this case is the sum of the numbers between lower bound a and an upper bound $a+n$. There are no restrictions on a, so it may be greater than, less than, or equal to zero. Let the value of the series be T such that

$$T = \sum_{j=a}^{a+n} j. \tag{5}$$

Now let $j \to i + a$, rearranging the lower limit and subtracting a from the upper limit such that

$$T = \sum_{i=0}^{n} (i + a). \tag{6}$$

Distributing the sum

$$T = \sum_{i=0}^{n} i + \sum_{i=0}^{a} a, \tag{7}$$

recognizing that the second sum is simply repeated addition of a constant $n+1$ times (and being careful not to forget the fact that the zero index still contributes)

$$T = \left(\sum_{i=0}^{n} i \right) + a(n+1). \tag{8}$$

Finally, recognizing that the remaining sum is equivalent to Gauss's sum since the first term (as written) is zero

$$T = \left(\sum_{i=1}^{n} i \right) + a(n+1). \tag{9}$$

Now, the special "boundary case" to be employed is when $-a = a + n$. This corresponds to the case when the upper and lower bound of the series in Equation (5) of equal magnitude but opposite sign, such that the sum is symmetric about zero. This symmetry about zero means $T = 0$ when $-a = a + n$ (equivalently the more useful form $a = -\frac{n}{2}$). Inserting $T = 0$ and $a = -\frac{n}{2}$ into Equation (9)

$$0 = \left(\sum_{i=1}^{n} i \right) - \frac{n}{2}(n+1), \tag{10}$$

and after simple rearrangement one recovers Gauss's answer

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}. \tag{11}$$

One possible advantage to this method is that with Gauss's sum solved, one has almost automatically solved for T with general a, n (all that remains is to substitute Gauss's result into Equation (9)).

[2] If anything this shows the superiority of Gauss's method, which achieves the same result using less machinery.